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VOLUME GROWTH AND ENTROPY

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ABSTRACT
An inequality is proved, bounding the growth rates of the volumes of iterates of smooth submanifolds in terms of the topological entropy. For C^∞-smooth mappings this inequality implies the entropy conjecture, and, together with the opposite inequality, obtained by S. Newhouse, proves the coincidence of the growth rate of volumes and the topological entropy, as well as the upper semicontinuity of the entropy.

1. Introduction

Let N be a compact m-dimensional C^∞-smooth manifold with some fixed Riemannian metric w. Let $f: N \to N$ be a continuous mapping. We recall the definition of the topological entropy of f.

Let d be the metric on N induced by w. For $n = 0, 1, \ldots$, define a new metric $d_{f,n}$ on N by

$$d_{f,n}(x, y) = \max_{i=0,1,\ldots,n} d(f^i(x), f^i(y)).$$

Denote by $M_f(n, \varepsilon)$ the minimal number of ε-balls in the $d_{f,n}$-metric, covering N. Then the topological entropy $h(f)$ can be defined by

$$h(f) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log_2 M_f(n, \varepsilon).$$

(See [1], [3].)

Let $S_l(f)$, $l = 0, 1, \ldots, m$, denote the logarithm of the spectral radius of $f_*: H_l(N, \mathbf{R}) \to H_l(N, \mathbf{R})$, $S(f) = \max_l S_l(f)$.

The entropy conjecture (in its first version) is that for f C^1-smooth, $S(f) \leqq h(f)$. (See [9], [5], and a recent paper [4], where a unified version of this conjecture is posed.)

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We prove here the following result:

THEOREM 1.1. *For f C^∞-smooth, $S(f) \leqq h(f)$.*

In fact we prove that for C^∞-smooth f, the entropy $h(f)$ bounds the growth rate of volumes, which, in turn, dominates any kind of homological growth.

For $k < \infty$ the "remainder terms" appear in the volume estimates. However, the resulting lower bounds for the entropy are nontrivial for any $k > 1$.

To state the results precisely we need some additional invariants of the dynamics of f. We define them assuming that f is C^1-smooth, although in fact one needs much less.

DEFINITION 1.2. For $f: N \to N$, $f \in C^1$,

$$R(f) = \lim_{n\to\infty} \frac{1}{n} \log \max_{x\in N} \| df^n(x) \|,$$

where the norm on the tangent bundle of N is given by w.

Let $\sigma: \mathbf{Q}^l \to N$ be a C^k-mapping, $k \geqq 1$, where $\mathbf{Q}^l = [0,1]^l \subseteq \mathbf{R}$. We denote the set of such σ by $\Sigma(k, l)$. $v(\sigma)$ is defined as the l-dimensional volume of the image of σ in N, counted with multiplicities:

$$v(\sigma) = \int_{\mathbf{Q}^l} \bar{v}(d\sigma),$$

where $\bar{v}(d\sigma)$ is the volume form, induced on $\mathbf{Q}^l$ by σ from the metric w on N.

Let, for a natural n,

$$v(f, \sigma, n) = v(f^n \circ \sigma).$$

DEFINITION 1.3. For $k \geqq 1$, $l \leqq m$,

$$v_{l,k}(f) = \sup_{\sigma\in\Sigma(l,k)} \overline{\lim_{n\to\infty}} \frac{1}{n} \log v(f, \sigma, n),$$

$$v_k(f) = \max_l v_{l,k}(f), \qquad \text{and}$$

$$v(f) = v_\infty(f).$$

Clearly, the invariants $h(f)$, $S(f)$, $R(f)$, $v_{l,k}(f)$ are independent of the choice of the metric w. $h(f)$ and $S_l(f)$ are topologically invariant, $R(f)$ and $v_{l,k}(f)$ are C^1- and C^k-invariant, respectively.

The following relations between the considered invariants are more or less immediate for $f \in C^1$:

(1) $h(f) \leqq mR(f)$,

(2) $v_{l,k}(f) \leqq lR(f)$,

(3) $S_l(f) \leqq v_{l,k}(f)$ for any $k = 1, \dots, \infty$.

The last inequality follows from the fact that the norm of the homology class is bounded by integrals of some fixed differential forms on this class; these integrals, in turn, are bounded by the volume of the chain, representing this class.

The following inequality was obtained by S. Newhouse [8]:

$$\text{For } f \in C^{1+\varepsilon}, \quad \varepsilon > 0, \quad h(f) \leqq v(f).$$

In the present paper the opposite inequality is proved:

THEOREM 1.4. *For* $f \in C^k$, $k = 1, \dots, \infty$, $l \leqq m$,

$$v_{l,k}(f) \leqq h(f) + \frac{2l}{k} R(f).$$

REMARK. In [12] we prove this inequality with l/k instead of $2l/k$. The examples below show this improved inequality is sharp.

By (1) and (2) above this inequality can be nontrivial for any $k > 2$.

By (3) and by the Newhouse's inequality, we get

COROLLARY 1.5. *For* $f \in C^k$, $k = 1, \dots, \infty$,

(i) $h(f) \leqq v(f) \leqq h(f) + (2m/k)R(f)$,

(ii) $S_l(f) \leqq h(f) + (2l/k)R(f)$,

(iii) $S(f) \leqq h(f) + (2m/k)R(f)$.

For f a diffeomorphism we can replace $2l$ in (ii) by $2\min(l, m-l)$, and respectively in (iii).

COROLLARY 1.6. *For* $f \in C^\infty$, $S(f) \leqq v(f) = h(f)$.

Let us introduce an additional invariant which measures the growth of the volume of the part of σ, which remains in small balls under all the iterations. Let $\tilde{\mathscr{B}}(n, \varepsilon)$ denote the set of ε-balls in the metric $d_{f,n}$.

Let $\sigma \in \Sigma(l, k)$. For any $S \subseteq \mathbf{Q}^l$ define $v(\sigma, S)$ as $\int_S v(d\sigma)$, and $v(f, \sigma, n, S)$ as $v(f^n \circ \sigma, S)$.

DEFINITION 1.7. For $\sigma \in \Sigma(k, l)$ and $\varepsilon > 0$, let

$$v^0(f, \sigma, n, \varepsilon) = \sup_{\mathscr{B} \in \tilde{\mathscr{B}}(n, \varepsilon)} v(f, \sigma, n, \sigma^{-1}(\mathscr{B})).$$

$v^0_{l,k}(f, \varepsilon)$ is defined as

$$\sup_{\sigma \in \Sigma(l,k)} \overline{\lim_{n \to \infty}} \frac{1}{n} \log^+ v^0(f, \sigma, n, \varepsilon), \quad \text{where } \log^+ a = \begin{cases} \log a, & a \geqq 1, \\ 0, & a \leqq 1. \end{cases}$$

We define also $v^0_{l,k}(f)$ by

$$v^0_{l,k}(f) = \lim_{\varepsilon \to 0} v^0_{l,k}(f, \varepsilon).$$

In fact, Theorem 1.4 follows from the estimate of $v^0_{l,k}(f)$:

THEOREM 1.8. *For $f \in C^k$, $k = 1, \dots, \infty$*

$$v^0_{l,k}(f) \leqq \frac{2l}{k} R(f).$$

In particular, for $f \in C^\infty$,

$$v^0_{l,\infty}(f) = 0, \qquad l \leqq m.$$

S. Newhouse proved (to appear) that the equalities $v^0_{l,\infty}(f) = 0$, $l \leqq m$, imply the upper semicontinuity of the topological and metric entropy in the space of C^∞-smooth mappings $f: N \to N$.

In this paper we prove the following result, which also implies the upper semicontinuity of $h(f)$ in the C^∞ case:

THEOREM 1.9. *For $f \in C^k$ and $g \to f$ in C^k-topology,*

$$\overline{\lim}\, h(g) \leqq h(f) + \frac{2m}{k} R(f).$$

REMARK 1. The bounds $v_{l,k}(f) \leqq h(f) + (l/k)R(f)$, where $2l$ of Theorem 1.4 is replaced by l, are sharp. The examples (essentially due to G. Margulis) are the following: for any $\xi > 0$ we can find a C^k-function g on $[0,1]$, such that g has zeroes at $x_1 < x_2 < \cdots$ and has local extrema at y_i, $x_i < y_i < x_{i+1}$, with $g(y_i) = (1/i)^{k+\xi}$.

Consider now the C^∞-smooth mapping $f: S^{2l} \to S^{2l}$, which in local coordinates at some fixed point is given by the linear transformation

$$l\Big\{\begin{pmatrix} \lambda & & & & & 0 \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \frac{1}{\lambda} & & \\ & 0 & & & \ddots & \\ & & & & & \frac{1}{\lambda} \end{pmatrix}, \qquad \lambda > 1,$$

and is extended to S^{2l} in such a way that $h(f) = 0$.

Consider now $\sigma \in \Sigma(l, k)$, $\sigma : \mathbf{Q}^l \to S^{2l}$, given in the above chart by

$$\sigma(t_1, \dots, t_l) = (t_1, \dots, t_l, g(t_1), \dots, g(t_l)).$$

Then one can easily verify that $v(g, \sigma, n)$ and, in fact, $v^0(f, \sigma, n, \varepsilon)$ for any $\varepsilon > 0$, is of order $\lambda^{l \cdot n/(k+\xi)}$. Hence

$$v_{l,k}(f) \geqq v^0_{l,k}(f) \geqq \frac{l}{k+\xi} \log \lambda = \frac{l}{k+\xi} R(f).$$

Of course, one can transform this example to have $\sigma \in \Sigma(l, \infty)$, but $f \in C^k$. For $l = 1$, the proof of Theorem 1.4 gives, in fact, this sharp bound.

REMARK 2. The proof of Theorem 1.4 allows one to bound the growth not only of the volume, but of various invariants, measuring complexity of submanifolds. In particular, for the so-called multidimensional variations (see [10], [6]), which seem to be useful in study of the entropy of mappings, it can be done by combining Theorem 2.1 below with the bounds of [11]. The detailed presentation of these results will appear separately.

2. The following result, which concerns the structure of differentiable mappings and does not involve iterations, is, in fact, the main result of this paper:

THEOREM 2.1. *Let B and B' denote the closed balls of radii 1 and 2, respectively, centered at $0 \in \mathbf{R}^m$. Let $f: B' \to \mathbf{R}^m$ be a C^k-mapping, $1 \leqq k < \infty$, such that*

$$\max_{x \in B'} \| d^s f(x) \| \leqq M, \qquad s = 1, \dots, k.$$

Let $\sigma : \mathbf{Q}^l \to B'$ be a C^k-mapping, such that $\max_{x \in \mathbf{Q}^l} \| d^s \sigma(x) \| \leqq 1$, $s = 1, \dots, k$.

Then there exist not more than $\kappa = \kappa(k, m, l, M) = \mu(k, m, l)(\log M)^{\nu(k,m,l)} \cdot M^{2l/k}$ mappings $\psi_j : \mathbf{Q}^l \to \mathbf{Q}^l$ with the following properties:

(1) ψ_j maps $\mathbf{Q}^l$ diffeomorphically onto its image, $j = 1, \dots, \kappa$.

(2) $\bigcup_{j=1}^{\kappa} \mathrm{Im}(\psi_j) \supseteq S$, where $S = (f \circ \sigma)^{-1}(B)$ is the set of points $x \in \mathbf{Q}^l$, such that $f \circ \sigma(x) \in B$.

(3) $\mathrm{Im}(f \circ \sigma \circ \psi_j) \subseteq B'$, $j = 1, \dots, \kappa$.

(4) $\max_{x \in \mathbf{Q}^l} \| d^s (f \circ \sigma \circ \psi_j(x)) \| \leqq 1$, $s = 1, \dots, k$, $j = 1, \dots, \kappa$.

Here the constants $\mu = \mu(k, m, l)$ and $\nu = \nu(k, m, l)$ depend only on k, m and l.

We prove Theorem 2.1 in Sections 3 and 4 below. In this section we use it to prove the results, stated in the introduction.

Let N be as above. We assume in addition that some finite C^∞-atlas $\Omega = (\Omega_p, \omega_p)$ is fixed, with Ω_p-open domains, covering N and $\omega_p : \Omega_p \to \mathbf{R}^m$ diffeomorphisms onto the images, such that for any $k \geqq 1$ all the k-th derivatives of $\omega_{p'} \circ \omega_p^{-1}$ are uniformly bounded, whenever defined.

Let $\alpha \geqq 1$ be a constant such that on any Ω_p the metric δ_p, induced by ω_p, satisfies

$$\frac{1}{\alpha}\,\delta_p \leqq d \leqq \alpha\delta_p.$$

Now let $f: N \to N$ be a C^k-mapping, $k \geqq 1$. We define $M_s(f)$, $s = 1, \dots, k$, as the maximum of $\| d^s f \|$ with respect to all the points in N and all the charts in Ω. We denote $M_1(f)$ by M.

Now we define $\varepsilon_0 > 0$, depending on N, the metric w, the atlas Ω and f, to be the maximum of real numbers ε, satisfying the following conditions:

(1) For any $x \in N$ the ball of the radius $10\,M\alpha^2\varepsilon$ in metric d, centered at x, is contained in some chart Ω_p of Ω.

(2) For any $s = 2, \dots, k$,

$$(3\alpha\varepsilon)^{s-1} M_s(f) \leqq M.$$

Now let $\sigma \in \Sigma(k, l)$, $\sigma: \mathbf{Q}^l \to N$.

PROPOSITION 2.2. *For any $\varepsilon < \varepsilon_0$,*

$$v^0(f, \sigma, n, \varepsilon) \leqq c(\sigma, \varepsilon) \cdot \kappa^n,$$

where $c(\sigma, \varepsilon)$ is the constant, depending only on σ and ε, and $v^0(f, \sigma, n, \varepsilon)$ and $\kappa = \kappa(k, m, l, M)$ are defined in Definition 1.7 and Theorem 2.1, respectively.

PROOF. Let a natural n be fixed and let $\mathcal{B}$ be some ball of radius ε in the metric $d_{f,n}$, centered at $x_0 \in N$. Let $x_i = f^i(x_0)$, $i = 0, 1, \dots, n$. For each $i = 0, \dots, n$ we fix some chart Ω_{p_i}, containing x_i together with the ball of the radius $10\,M\alpha^2\varepsilon$ in metric d, centered at x_i.

Let $\tilde{B}_i$ denote the $\alpha\varepsilon$-ball in metric δ_{p_i}, centered at x_i. By definition of the constant α, $\tilde{B}_i$ contains the ε-ball in metric d at x_i.

Hence the set $\tilde{\mathcal{B}} = \{x \in N,\ f^i(x) \in \tilde{B}^i,\ i = 0, \dots, n\}$ contains the ball $\mathcal{B}$, and therefore

$$v(f, \sigma, n, \mathcal{B}) \leqq v(f, \sigma, n, \tilde{\mathcal{B}}).$$

We note also that each $\tilde{B}_i$ is mapped by f into $\Omega_{p_{i+1}}$. Indeed, by definition of constant $M = M_1(f)$, $f(\tilde{B}_i)$ is contained in the $M\alpha^2\varepsilon$-ball in metric d at x_{i+1}, which belongs entirely to $\Omega_{p_{i+1}}$. The same is valid for the balls of doubled radius.

Let B and B' be, as above, the 1- and 2-balls, respectively, centered at $0 \in \mathbf{R}^m$. Define $f_i: B' \to \mathbf{R}^m$ by

$$f_i = \left\langle \frac{1}{\alpha\varepsilon} \right\rangle \circ \omega_{p_i} \circ f \circ \omega_{p_{i-1}}^{-1} \circ \langle \alpha\varepsilon \rangle, \qquad i = 1, \dots, n,$$

where $\langle \beta \rangle$ denotes the linear mapping $x \to \beta x$ of $\mathbf{R}^m$.

By the assumptions above, f_i are well-defined and satisfy the following conditions:

(1) $\max_{x \in B'} \| d^s f_i(x) \| \leqq (\alpha\varepsilon)^{s-1} M_s(f) \leqq M$, $s = 1, \dots, k$.

(2) Let $S_i = \{x \in \mathbf{R}^m,\ F_j(x) \in B,\ j = 0, 1, \dots, i\}$, where $F_j = f_j \circ \cdots \circ f_1$, $j = 1, \dots, n$, $F_0 = \mathrm{Id}$. Then

$$\tilde{\mathscr{B}} = \omega_{p_0}^{-1} \circ \langle \alpha\varepsilon \rangle (S_n).$$

LEMMA 2.3. *Let $\tilde{\sigma}: \mathbf{Q}^l \to B'$ be a C^k-mapping, satisfying $\| d^s \tilde{\sigma} \| \leqq 1$, $s = 1, \dots, k$. Then for $i = 0, \dots, n$, there exist not more than κ^i diffeomorphisms $\psi_{ij}: \mathbf{Q}^l \to \mathbf{Q}^l$, $j = 1, \dots, \kappa^i$ such that*

(1) $\| d^s (F_i \circ \tilde{\sigma} \circ \psi_{ij}) \| \leqq 1$, $s = 1, \dots, k$,

(2) $\mathrm{Im}(F_i \circ \tilde{\sigma} \circ \psi_{ij}) \subseteq B'$,

(3) $\sigma^{-1}(S_i) \subseteq \bigcup_{j=1}^{\kappa^i} \mathrm{Im}(\psi_{ij})$.

PROOF. By induction. For $i = 0$ $\tilde{\sigma}$ itself satisfies the conditions of the lemma. Assume that for $p \geqq 0$ the lemma is proved.

Denote by σ_j the mappings

$$F_p \circ \tilde{\sigma} \circ \psi_{pj}: \mathbf{Q}^l \to \mathbf{R}^m.$$

Each σ_j, and the mapping $f_{p+1}: B' \to \mathbf{R}^m$, satisfy the assumptions of Theorem 2.1. Hence, by this theorem, for any σ_j we can find not more than κ mappings $\psi_q: \mathbf{Q}^l \to \mathbf{Q}^l$ such that $\| d^s (f_{p+1} \circ \sigma_j \circ \psi_q) \| \leqq 1$, $s = 1, \dots, k$, $\mathrm{Im}(f_{p+1} \circ \sigma_j \circ \psi_q) \subseteq B'$, and the images of ψ_q cover $(f_{p+1} \circ \sigma_j)^{-1}(B) = (\sigma \circ \psi_{p,j})^{-1}(S_{p+1})$.

But then we have $f_{p+1} \circ \sigma_j \circ \psi_q = F_{p+1} \circ \sigma \circ \psi_{pj} \circ \psi_q$, and hence the mappings $\psi_{p+1,\alpha}: \mathbf{Q}^l \to \mathbf{Q}^l$, $\psi_{p+1,\alpha} = \psi_{p,j} \circ \psi_q$, where $\alpha = (j, q)$, satisfy the conditions of the lemma, and their number does not exceed $\kappa^p \cdot \kappa = \kappa^{p+1}$.

Lemma 2.3 is proved.

To complete the proof of Proposition 2.2 it is enough to note that we can subdivide $\mathbf{Q}^l$ into $C(\sigma, \varepsilon)$ subcubes $\mathbf{Q}$ and reparametrize them linearly by

$h_\beta: \mathbf{Q}^l \to \mathbf{Q}^l \subseteq \mathbf{Q}^l$ in such a way that the mappings

$$\left\langle \frac{1}{\alpha\varepsilon} \right\rangle \circ \omega_{p_0} \circ \sigma \circ h_\beta : \mathbf{Q}^l \to \mathbf{R}^m$$

have all the derivatives up to the k-th, bounded by 1, and those of their images, which intersect B, are contained in B'.

COROLLARY 2.4. *For any* $\varepsilon < \varepsilon_0$,

$$v^0_{l,k}(f,\varepsilon) \leqq \log \mu + \nu \log\log M + \frac{2l}{k} \log M.$$

PROOF. It follows immediately from the definition of $v^0_{l,k}(f,\varepsilon)$ and Proposition 2.2.

Now let $\xi > 0$ be given. Define $q(\xi)$ as the minimal integer, satisfying

$$\frac{1}{q}(\log \mu + \nu \log\log M) + \frac{\log q}{q}\nu \leqq \xi.$$

Let $\tilde{\varepsilon}(\xi) > 0$ be defined as ε_0 above, but for the mapping $f^{q(\xi)}$ instead of f.

THEOREM 2.5. *For any* $\varepsilon < \tilde{\varepsilon}(\xi)$,

$$v^0_{l,k}(f,\varepsilon) \leqq \xi + \frac{2l}{k} \cdot \frac{1}{q(\xi)} \log M_1(f^{q(\xi)}).$$

In particular,

$$v^0_{l,k}(f,\varepsilon) \leqq \xi + \frac{2l}{k} \log M.$$

PROOF. We apply Corollary 2.4 to the mapping

$$f^q, q = q(\xi): \ v^0_{l,k}(f^q,\varepsilon) \leqq \log \mu + \nu(\log q + \log\log M) + \frac{2l}{k} \log M_1(f^q).$$

(Indeed, $M_1(f^q) \leqq M^q$.)

But clearly $v^0_{l,k}(f^q,\varepsilon) \geqq q v^0_{l,k}(f,\varepsilon)$. Hence

$$v^0_{l,k}(f,\varepsilon) \leqq \frac{1}{q}[\log \mu + \nu(\log q + \log\log M)] + \frac{2l}{k}\frac{1}{q} \log M_1(f^q) \leqq \xi + \frac{2l}{k}\frac{1}{q} \log M_1(f^q).$$

Theorem 2.5 is proved.

Now we can prove Theorem 1.8. Indeed, taking $\varepsilon \to 0$, and hence $\xi \to 0$ and $q \to \infty$, we immediately obtain from Theorem 2.5:

$$v^0_{l,k}(f)=\lim_{\varepsilon\to 0} v^0_{l,k}(f,\varepsilon)\leqq \frac{2l}{k}\lim_{q\to\infty}\frac{1}{q}\log M_1(f^q)=\frac{2l}{k}R(f).$$

PROOF OF THEOREM 1.4. Let $M_f(n,\varepsilon)$ denote, as above, the minimal number of ε-balls in the $d_{f,n}$-metric, covering N. Then for any $\sigma\in\Sigma(l,k)$, and for any $\varepsilon>0$, $v(f,\sigma,n)\leqq M_f(n,\varepsilon)\cdot v^0(f,\sigma,n,\varepsilon)$. Hence for each $\varepsilon>0$, $v_{l,k}(f)\leqq h(f)+v^0_{l,k}(f,\varepsilon)$. Taking limit as $\varepsilon\to 0$, we get

$$v_{l,k}(f)\leqq h(f)+v^0_{l,k}(f)\leqq h(f)+\frac{2l}{k}R(f).$$

PROOF OF THEOREM 1.9. In fact, we prove a more precise statement.

Let $f\colon N\to N$ be a continuous mapping. Let $W=W(f,k,M,M_2,\dots,M_k,\delta)$ denote the set of C^k-smooth $g\colon N\to N$, satisfying $M_1(g)\leqq M$, $M_s(g)\leqq M_s$, $s=2,\dots,k$, and $d(f,g)\leqq\delta$, where $d(f,g)=\max_{x\in N}d(f(x),g(x))$.

THEOREM 2.6.

$$\limsup_{\delta\to 0}\sup_{g\in W} h(g)\leqq h(f)+\frac{2m}{k}\log M.$$

PROOF. (1) We choose $\varepsilon_0>0$ so that the conditions on the choice of ε_0, described above, are satisfied for any $g\in W$ (with $\delta=1$).

(2) For a given $\xi>0$, we fix a natural p such that $M_f(p,\frac{1}{2}\varepsilon_0)\leqq 2^{(h(f)+\xi)p}$.

(3) We take $\delta>0$ so small that for any $g\in W$, any $\frac{1}{2}\varepsilon_0$-ball in the $d_{f,p}$-metric is contained in some ε_0-ball in the $d_{g,p}$-metric. Hence for any $g\in W$,

$$M_g(p,\varepsilon_0)\leqq M_f(p,\tfrac{1}{2}\varepsilon_0)\leqq 2^{(h(f)+\xi)p}$$

(4) Now for any $\sigma\in\Sigma(l,k)$, by the construction above, $\mathbf{Q}^l$ can be covered by at most

$$c(\sigma,\varepsilon_0)\chi=c(\sigma,\varepsilon_0)\cdot M_g(p,\varepsilon_0)\kappa^p\leqq c(\sigma,\varepsilon_0)2^{(h(f)+\xi)p}\kappa^p$$

mappings $\psi_j\colon \mathbf{Q}^l\to\mathbf{Q}^l$, such that $\|d^s(g^p\circ\sigma\circ\psi_j)\|\leqq 1$, $s=1,\dots,k$.

By induction, for $n=p\cdot i$, $\mathbf{Q}^l$ can be covered by at most $c(\sigma,\varepsilon_0)\chi^i$ such mappings. Hence

$$v(g,\sigma,n)\leqq c(\sigma,\varepsilon_0)2^{(h(f)+\xi)n}\kappa^n, \qquad\text{and}$$

$$v_{l,k}(g)\leqq h(f)+\xi+\log\kappa.$$

Denoting $\lim_{\delta\to 0}\sup_{g\in W}v_{l,k}(g)$ by $\tilde v_{l,k}$, we obtain $\tilde v_{l,k}\leqq h(f)+\log\kappa$. Finally, replacing f and g by f^q and g^q, and taking $q\to\infty$, we get

$$\tilde v_{l,k}\leqq h(f)+\frac{2l}{k}\cdot\log M.$$

Since by S. Newhouse's inequality, $\lim_{\delta\to 0}\sup_{g\in W} h(g) \leqq \tilde{v}_{l,k}$, Theorem 2.6 is proved.

Note that in the case $g \in W$, $g \to f$ in C^1 we can in κ replace M by $M_1(f)$, and hence in the final formula $\log M$ by $R(f)$. This proves Theorem 1.9.

REMARK 1. Of course, if $f: N \to N$ can be approximated in C^0-topology by C^k-smooth mappings with uniformly bounded derivatives, then, in fact, $f \in C^{k-1}$.

REMARK 2. One can easily given more "effective" versions of Theorem 2.6, similar to those of Theorem 2.5. The only information one needs to give δ explicitly, is the behavior of $M_f(n, \varepsilon)$.

3. Proof of Theorem 2.1

First we estimate the derivatives of $f \circ \sigma$. We have

$$d^s(f\circ\sigma)(x) = \sum_{j=1}^{s} d^j f(\sigma(x)) \circ P_j(d\sigma(x)), \qquad s = 1, \dots, k,$$

where $P_j(d\sigma)$ are the universal (depending only on m, l, s and j) polynomials in partial derivatives of σ. Hence, using the assumptions on the derivatives of f and σ, we obtain

$$\| d^s(f\circ\sigma)(x)\| \leqq \sum_{j=1}^{s} \| d^j f(\sigma(x))\| \cdot \| P_j(d\sigma(x))\|$$

$$\leqq M \sum_{j=1}^{s} \| P_j(d\sigma(x))\| \leqq Mc(m, l, s, j) \leqq Mc_1(m, l, s) = Mc_1.$$

From now on we proceed as follows: we subdivide $\mathbf{Q}^l$ into smaller parts and then reparametrize each part by the same unit cube $\mathbf{Q}^l$. At every step, except one, the subdivisions and the corresponding reparametrizations will be linear, with the norm at most 1.

(1) Let $\gamma = 2c_1([M]+1)^{1/k}([\sqrt{l}]+1)$. We subdivide $\mathbf{Q}^l$ into subcubes $\mathbf{Q}^l_i$ of size $1/\gamma$, $i = 1, \dots, \kappa_1$, where $\kappa_1 = \gamma^l = c_2 \cdot M^{l/k}$.

Let $\varphi^1_i: \mathbf{Q}^l \to \mathbf{Q}^l_i$ be the affine isomorphism of the form $x = \varphi^1_i(x') = \bar{x}_i + (1/\gamma)x'$.

Then for any $s = 1, \dots, k$, and for $g_i = f \circ \sigma \circ \varphi^1_i$ we have

$$\| d^s g_i \| \leqq \left(\frac{1}{\gamma}\right)^s \| d^s(f\circ\sigma)\| \leqq \left(\frac{1}{\gamma}\right)^s c_1 M \leqq \tfrac{1}{2}(1/\sqrt{l})^s M^{1-s/k}.$$

In particular, $\|d^s g_i\| \leqq M$, $s = 1, \ldots, k-1$, and

$$\|d^k g_i\| \leqq \tfrac{1}{2}(1/\sqrt{l})^k.$$

(2) We continue to work with one of the mappings g_i, which we denote by g.

Let $p : \mathbf{R}^l \to \mathbf{R}^m$ be the Taylor polynomial of g of degree k at the center of $\mathbf{Q}^l$, $p = (p_1, \ldots, p_m)$ with $p_j(x_1, \ldots, x)$-polynomials of degree k.

By the Taylor formula:

$$(*) \qquad \max_{x \in \mathbf{Q}} \|g(x) - p(x)\| \leqq \frac{1}{k!}\left(\frac{\sqrt{l}}{2}\right)^k \max \|d^k g\| \leqq \tfrac{1}{4}.$$

(3) At this stage we use for the first time the fact that we need to cover only the part of $\mathbf{Q}^l$ which is mapped by g into B. So let $\tilde{S} = g^{-1}(B) \subseteq \mathbf{Q}^l$. By (*)

$$\tilde{S} \subseteq S' = \{x \in \mathbf{Q}^l \mid \|p(x)\| \leqq \tfrac{5}{4}\}.$$

S' is a semialgebraic subset in $\mathbf{Q}^l$ (see e.g. [2]), defined by the only polynomial inequality of degree $2k$:

$$\|p(x)\|^2 = \sum_{r=1}^{m} p_r^2(x) \leqq \frac{25}{16}.$$

(4) We use now the following proposition (entirely belonging to real semialgebraic geometry), whose proof for $l = 1, 2$ will be given in Section 4:

PROPOSITION 3.1. *Let $A \subseteq \mathbf{Q}^l$ be a semialgebraic set in $\mathbf{Q}$, defined by an inequality $h \geqq 0$, with h a polynomial of degree d.*

Let $C > 0$ and a natural k be given.

Then A can be subdivided into not more than $\kappa_2 = \kappa_2(d, k, l, C) = c'(\log C)^{c''}$ closed semialgebraic subsets A_i with the following property:

For any $i = 1, \ldots, \kappa_2$, either

(a) *A_i is contained in some subcube $\mathbf{Q}_i^l$ of $\mathbf{Q}^l$ of size $1/C$, or*

(b) *There exists a semialgebraic and C^k-smooth diffeomorphism $\varphi_i : \mathbf{Q}^l \to A_i$ of the unit cube onto A_i with*

$$\|d^s \varphi_i\| \leqq 1, \qquad s = 1, \ldots, k.$$

The constants c' and c'' here depend only on l, d and k.

(For definitions and basic properties of semialgebraic sets and mappings see e.g. [2].)

(5) We apply Proposition 3.1 to the semialgebraic set S', defined above, with $C = \sqrt{l}M$, k as above and $d = 2k$.

Let S'_i, $i=1,\dots,\kappa_2$, be the parts of S', given by Proposition 3.1, $\kappa_2 = c_3(k,l)(\log M)^{c_4(k,l)}$.

Consider first S', which are contained in subcubes $\mathbf{Q}^l_\alpha \subseteq \mathbf{Q}^l$ of size $1/2\sqrt{l}M$. Using the affine reparametrizations $\tilde{\varphi}_\alpha : \mathbf{Q}^l \to \mathbf{Q}^l_\alpha$, we obtain

$$\| d^s(g \circ \tilde{\varphi}_\alpha)\| \leqq \left(\frac{1}{2M\sqrt{l}}\right)^s \| d^s g\| \leqq \left(\frac{1}{2\sqrt{l}}\right)^s.$$

In particular, $\| d(g \circ \tilde{\varphi}_\alpha)\| \leqq 1/2\sqrt{l}$, and hence $\mathrm{Im}(g \circ \tilde{\varphi}_\alpha) \subseteq B'$. Thus all the conditions of Theorem 2.1 are satisfied for the mappings $\psi_j = \varphi^1_i \circ \tilde{\varphi}_\alpha : \mathbf{Q}^l \to \mathbf{Q}^l$.

(6) Now consider some part S'_β of S' of the second type: S'_β is the image of C^k-diffeomorphism φ^2_β: $\mathbf{Q}^l \to \mathbf{Q}^l$ with

$$\| d^s \varphi^2_\beta \| \leqq 1, \qquad s = 1,\dots,k.$$

We have for any $x \in \mathbf{Q}^l$ and $g' = g \circ \varphi^2_\beta$: $\mathbf{Q}^l \to \mathbf{R}^m$:

(a) $\| g'(x)\| \leqq \frac{3}{2}$.

Indeed, $g'(x) = g(y)$, where $y = \varphi^2_\beta(x) \in S'$. Then $\| p(y)\| \leqq \frac{5}{4}$, and since $\| g - p\| \leqq \frac{1}{4}$, $g(y) \leqq \frac{3}{2}$.

(b) $\| d^s g'\| \leqq \Sigma^s_{j=1} \| d^j g\| \cdot \| P_j(d\varphi^2_\beta)\| \leqq c_5(k,l,m) \cdot M$.

The main advantage we have now is that in contrast to the initial situation, the norm of g' is now uniformly bounded by $\frac{3}{2}$ on all the cube $\mathbf{Q}^l$. This allows us to use the estimates for the "intermediate" derivatives as follows:

(7) Subdivide $\mathbf{Q}^l$ into subcubes $\mathbf{Q}^l_\eta$ of size $1/\gamma'$, where $\gamma' = c_5([M]+1)^{1/k}$, $\eta = 1,\dots,\kappa_3$, with $\kappa_3 = \gamma'^l = c_6 M^{l/k}$.

As above, after reparametrization by φ^3_η: $\mathbf{Q}^l \to \mathbf{Q}^l$ we have for $g'' = g' \circ \varphi^3_\eta$: $\mathbf{Q}^l \to \mathbf{R}^m$:

$$\| g''\| \leqq \tfrac{3}{2}, \quad \| d^k g''\| \leqq 1 \qquad \text{on all the } \mathbf{Q}^l.$$

(8) Now we prove the lemma, bounding intermediate derivatives through the function itself and its highest derivative.

LEMMA 3.2. *Let $h: \mathbf{Q}^l \to \mathbf{R}^m$ be a C^k-mapping with $\| h(x)\| \leqq C_1$, $\| d^k h(x)\| \leqq C_2$ for any $x \in \mathbf{Q}$.*

Then $\| d^s h(x)\| \leqq c_7(k,l,m)(C_1 + C_2)$, $x \in \mathbf{Q}^l$, $s = 1,\dots,k-1$.

PROOF. Let p be the Taylor polynomial of h of degree k at the center of $\mathbf{Q}^l$. Then

$$(*) \qquad \| d^s h(x) - d^s p(x)\| \leqq c'_7(k,l,m) C_2, \qquad x \in \mathbf{Q}^l, \quad s = 0,1,\dots,k-1.$$

In particular, $\| h - p\| \leqq c'_7 \cdot C_2$, and hence

$$\|p(x)\| \leqq c_7' C_2 + C_1, \qquad x \in \mathbf{Q}^l.$$

Now by the Markov inequality (see [7]) for the polynomial p of degree k one has

$$\|d^s p(x)\| \leqq c_7'' \max_{x \in \mathbf{Q}^l} \|p(x)\| \leqq c_7''(c_7' C_2 + C_1),$$

$s = 1, \dots, k-1$, $x \in \mathbf{Q}^l$. Using once more (*), we obtain

$$\|d^s h(x)\| \leqq c_7''(c_7' C_2 + C_1) + c_7' C_2 \leqq c_7(C_1 + C_2),$$

$s = 1, \dots, k-1$, $x \in \mathbf{Q}^l$. Lemma 3.2 is proved.

Applying this lemma to the mapping g'', we get $\|d^s g''(x)\| \leqq c_8(k, l, m)$.

(9) Subdividing $\mathbf{Q}^l$ once more into the subcubes $\mathbf{Q}^l_\rho$ of size $1/c_8$, $\rho = 1, \dots, \kappa_4 = c_8^l$, and reparametrizing by φ^4_ρ: $\mathbf{Q}^l \to \mathbf{Q}^l_\rho$, we get finally

$$\|d^s(g'' \circ \varphi^4_\rho)\| \leqq 1, \qquad s = 1, \dots, k.$$

Thus we put $\psi_j = \varphi^1_i \circ \varphi^2_\beta \circ \varphi^3_\eta \circ \varphi^4_\rho$, $j = (i, \beta, \eta, \rho)$. By the construction,

$$f \circ \sigma \circ \psi_j = f \circ \sigma \circ \varphi^1_i \circ \varphi^2_\beta \cdot \varphi^3_\eta \varphi^4_\rho = g_i \circ \varphi^2_\beta \circ \varphi^3_\eta \circ \varphi^4_\rho = g'_{i,\beta} \circ \varphi^3_\eta \circ \varphi^4_\rho = g''_{i,\beta,\eta} \circ \varphi^4_\rho$$

and hence its derivatives are bounded by 1, and its image is contained in B'.

By the construction of ψ_j, their images cover the set $S = (f \circ \sigma)^{-1}(B)$. The number of ψ_j does not exceed

$$\begin{aligned} \kappa &= \kappa_1 \cdot \kappa_2 \cdot \kappa_3 \cdot \kappa_4 \\ &= c_2 M^{l/k} \cdot c_3 (\log M)^{c_4} \cdot c_6 M^{l/k} \cdot c_8^l \\ &= \mu(k, l, m)(\log M)^{\nu(k,l,m)} \cdot M^{2l/k}. \end{aligned}$$

Theorem 2.1 is proved.

4. Proof of Proposition 3.1

We give here the proof only for $l = 1$ and 2. The proof in the general case will appear separately (see [12]). However, the two-dimensional case represents the main difficulties and ideas of the general situation.

For $l = 1$ the proof is very simple. We have a subset A in $[0, 1]$ defined by an inequality $h \geqq 0$, with h a polynomial of degree d. If $h \equiv 0$, then we take $[0, 1]$ as the only part A_1. Otherwise the equation $h = 0$ has at most d solutions in $[0, 1]$. Hence A consists of not more than $(d + 2)/2$ intervals, which we take as A_i.

Note that for $l = 1$ we do not need the additional parameter C, $\kappa_2(d, k, 1) = [(d+2)/2]$, and the reparametrizations φ_i are linear.

Let $l = 2$. We consider $\mathbf{Q}^2$ as the subset of $\mathbf{R}^2$, defined by $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. We can assume that $h \not\equiv 0$. Consider the curve

$$v' = \{(x, y) \in \mathbf{Q}^2,\ h(x, y) = 0\},$$

and let v be the corresponding reduced curve. Let $\sigma(v)$ be the set consisting of all the singular points of v, of the regular points of v, where the tangent line is parallel to Oy, and of all the intersection points of v with the boundary of $\mathbf{Q}^2$.

Then $\sigma(v)$ consists of the finite set $\sigma'(v) = \{(x_i, y_i),\ i = 1, \ldots, c_1(d)\}$ and, perhaps, of some vertical lines $x = x_j$, $j = c_1(d)+1, \ldots, c_2(d)$, and two horizontal lines $y = 0$ and $y = 1$.

Assume that $x_1 \leqq x_2 \leqq \cdots \leqq x_{c_2}(d)$ and consider two vertical lines $x = x_i$ and $x = x_{i+1}$, $x_i < x_{i+1}$. We denote $[x_i, x_{i+1}] \times [0, 1]$ by $\mathbf{Q}'$.

By construction, $v \cap \mathbf{Q}'$ consists of at most $c_3(d)$ segments v_j, which can be represented by $y = y_j(x)$, $x_i \leqq x \leqq x_{i+1}$, with $y_j(x)$ continuous on $[x_i, x_{i+1}]$, analytic on (x_i, x_{i+1}) and an algebraic function, satisfying $0 \leqq y_j(x) \leqq 1$.

We reparametrize $\mathbf{Q}'$ by the mapping

$$\psi: \mathbf{Q}^2 \to \mathbf{Q}', \qquad \psi(x, y) = (x_i + x(x_{i+1} - x_i), y).$$

Below we constantly use the reparametrizations of this form, not specifying their concrete expressions.

LEMMA 4.1. *Let $y = y(x)$ be an algebraic function (defined by an equation of degree d) on $[0, 1]$, which is continuous on $[0, 1]$, analytic on $(0, 1)$ and satisfies $0 \leqq y(x) \leqq 1$. Let $C > 0$ and a natural k be given. Then there exists a partition of $[0, 1]$ into not more than $\kappa'(d, k, C)$ subintervals I_i, such that either (1) $|I_i| \leqq 1/C$, or (2) for the affine reparametrization $\varphi_i: [0, 1] \to I_i$, all the derivatives of $y(\varphi_i(x))$ up to order k are bounded by 1.*

Here $\kappa'(d, k, C) = \bar{c}(d, k)(\log C)^k$.

PROOF. (a) In the course of the proof we will several times subdivide and reparametrize our interval. If at some stage we obtain an interval of length $\leqq 1/C$ (in a new reparametrization), we do not subdivide it more, since its image in the original interval has at most the same length and hence satisfies condition (1).

(b) Consider all the zeroes x_i in $(0, 1)$ of those of $d^s y/dx^s$, $s = 1, \ldots, k+1$, which do not vanish identically, $i = 1, \ldots, \hat{c}(d, k)$. Subdividing $[0, 1]$ by these points, we can assume that all the derivatives of y on $[0, 1]$, up to the k-th, do not change sign and are monotone.

Let, for example, $y' \geqq 0$ and increase. Since $0 \leqq y \leqq 1$, we have $\int_0^1 y'dx \leqq 1$. Consider the points $z_j = 1 - (\frac{1}{2})^j$, $j = 0, 1, \ldots, q = [\log C] + 1$. Denote I_j the interval $[z_{j-1}, z_j]$, $j = 1, \ldots, q$, $J = [z_q, 1]$.

We have $|J| = (\frac{1}{2})^q \leqq 1/C$, $|I_j| = (\frac{1}{2})^j$. Now $y'/I_j \leqq 2^j$. Indeed, if $y' > 2^j$ at some point of I_j, then, by monotonicity, $y' > 2^j$ on $[z_j, 1]$, and since the length of $[z_j, 1]$ is $(\frac{1}{2})^j$, this contradicts the inequality $\int_0^1 y' \leqq 1$.

Since $|I_j| = (\frac{1}{2})^j$, after a reparametrization of I_j by $[0, 1]$, we obtain a new function y with $0 \leqq \tilde{y}' \leqq 1$.

On each new interval we apply the same construction to the second derivative of y and so on up to the k-th derivative. Since the property of a monotonicity of the derivatives persists under linear reparametrizations, the total number of the subintervals we obtained is, at most, $\hat{c}(d, k)2^k(\log C)^k$.

Lemma 4.1 is proved.

Applying consequently this lemma to each of the algebraic functions y_j (and noticing that in our reparametrizations the derivatives of other y_i can only decrease), we reduce the situation to the following one: the set $A \subseteq \mathbf{Q}^2$ is given by the inequality $y_1(x) \leqq y \leqq y_2(x)$, where $y_1 \leqq y_2$ are two algebraic functions, analytic on $[0, 1]$ with all the derivatives up to the k-th bounded by 1.

Consider the diffeomorphism $\phi: \mathbf{Q}^2 \to A$ of $\mathbf{Q}^2$ onto A, given by

$$\phi(x, y) = (x, y_1(x) + y(y_2(x) - y_1(x))).$$

Clearly, all the partial derivatives of the components of ϕ (up to the k-th) are bounded by 2. Hence subdividing $\mathbf{Q}^2$ into $c_5(k)$ smaller subcubes of size depending only on k, one gets $\varphi_i: \mathbf{Q}^2 \to A_i$, satisfying $\|d^s\varphi_i\| \leqq 1$, $s = 1, \ldots, k$.

There remained some vertical strips of width $\leqq 1/C$. We apply to these strips the same construction, but now with horizontal lines instead of vertical ones. Finitely we cover by mappings φ_i all the set A, except some parts, contained in the subcubes of the size $1/C$. The total number of subsets A_i does not exceed, by the construction,

$$[c_2(d)(\kappa')^{c_3(d)} \cdot c_5(k)]^2 = (c_2(d)\bar{c}(d, k) \cdot c_5(k))^2(\log C)^{2k \cdot c_3(d)} = c'(d, k)(\log C)^{c''(d, k)}.$$

The case $l = 2$ of Proposition 3.1 is proved.

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